Enumeration of Relationships in Polyamorous Groups

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Abstract

In a LiveJournal post, Tacit et al [2006] proposed an equation to enumerate the 'total number of possible relationship configurations' for any group of polyamorous people. Unfortunately, the equation arrived at is incorrect. This incorrectness will be shown herein by mapping this family of problems to graph theory. Seven problem variants will be proposed, rigorously defined, and, for the first four, formulae provided to enumerate the graphs they describe. Results will be tabulated and exhaustive diagrams of the simpler graph orders will be provided to aid the reader in visualising the problem and solutions.

1 Introduction

In a LiveJournal post, Tacit et al [2006] proposed an equation to enumerate the 'total number of possible relationship configurations' for any group of polyamorous people. The equation, the original form of which is in equation 1 below, can be shown to be wrong. This document will show this by a combination of counter-example, reduction to a well-known problem and *reductio ad absurdum*, as aided by computer-based enumeration, visualisation and comparison.

The original, terse description of the problem seeks out 'the equation that will tell you for any size group of people n how many possible relationship configurations (couples, triads, and so on) are possible within that group' [Tacit et al, 2006].

It is easy to find the ambiguities in this problem statement. This document will attempt to resolve the ambiguities by defining a relationship in the context of this problem, thereby transferring this enumeration problem to graph theory. It will provide eight rigorously defined problem variants stemming from the original problem, and matching the spirit of the original problem. Solutions to these will be produced and verified by computer.

2 Reduction to Graph Theory

It is useful to visualise this problem in the realm of graph theory. To do this, it is helpful to define various concepts in both graph theory and polyamory, and also to link the two groups of concepts together where terms are used interchangeably.

Definition 1 ***Vertex*** *A person who may be involved in zero or more relationships.*

Definition 2 ***Relationship*** *A romantic link between two people.*

If a person is seen as the vertex of a graph, then a relationship is a link between two vertices, and, by definition, an edge. This allows problem components to be restated within graph theory.

It will be shown that the problems at hand are reducible to well-known problems in graph theory, and thereby possible to arrive at equally well-known (if non-trivial) answers.

Definition 3 ***Graph*** *A collection of vertices (people) and edges (relationships between them). This particular type of graph is an undirected one (there are no arrows from one person to another — relationships being equal in nature).*

Definition 4 ***Labelled Graph*** *A graph where the vertices (people) are named and not interchangeable. A graph of Alice, John and Bob, where Alice is in a relationship with both John and, Bob is labelled.*

Definition 5 ***Unlabelled Graph*** *A graph where the vertices are anonymous. It's used to study abstract properties of graphs. A vee is an unlabelled graph. The relationship o Alice, John and Bob above becomes the abstract topologial shape of a vee if you ignore their names.*

Definition 6 ***Connected Graph*** *A connected graph is one in which all vertices are, directly or indirectly, connected to all others.*

Definition 7 ***Fully-Connected Graph*** *A graph where each vertex is directly connected to every other vertex.*

Definition 8 ***Empty Graph*** *An empty graph has no relationships.*

Definition 9 ***Polyamorous Group*** *a group of two or more polyamorous people in zero or more relationships (this is purposefully congruent to the definition of a graph).*

Definition 10 ***Isolated vertex*** *an isolated or (appropriately) hermit vertex in a graph is one which is not connected to any others. The colloquial, mainstream term for a person who acts as an isolated vertex is 'single'.*

Definition 11 ***Vee*** *a three-person group in which two relationships exist, i.e. a three-vertex, two-edge graph.*

Definition 12 ***Triad*** *a three-person group in which every person is attached to every other, i.e. a fully-connected 3-vertex, 3-edge graph.*

Definition 13 ***Quad*** *a four-person group in which every person is attached to every other, i.e. a fully-connected 4-vertex, 6-edge graph.*

Definition 14 ***Quint*** *a five-person group in which every person is attached to every other, i.e. a fully-connected 5-vertex, 10-edge graph.*

3 Ambiguity in the Original Problem

A number of questions are raised by the way the original problem is stated. Some where answered in the original LiveJournal post, others seem to remain unanswered, and even unasked. These questions are designated as Q1 to Q4:

- **Q1** Does the problem also define as 'relationship configuration' a group of n, pairwise unrelated n (i.e single) people? The consensus seems to be that this should not be enumerated. This discussion, however, enumerates this case too, as other results are based on this.
- **Q1** Can single persons be part of a 'relationship configuration' along with non-singles? Id est, does the problem include scenarios such as a group of three people including a couple and a single person (three-vertex, one-edge unconnected graph)?

Figure 1: A two-vertex graph depicting two single people (left); and two related people (right).

- **Q3** Is a 'relationship configuration' defined as a single relationship only, or can it include a set of mutually unconnected relationships? This question seems to have not been discussed at all, although it is not necessarily excluded by the description of the problem.
- **Q4** Does 'relationship configuration' group topologically isomorphous relationships? Id est, are all three possible 'vees' in a three-person group considered *one* 'relationship configuration'? There is only one distinct topology for three people to be connected by two relationships (three-vertex, two-edge graphs), but there can be three distinct vees between individual groups of three people. In this document, it is assumed that this is not the case. This is based on the original formula, and on the description of the problem alluding to individual people. However, for completeness, three of the provided problem variants involve unlabelled graphs.

4 Dealing with the Ambiguity — Problem Variants

Seven individual problem variants are thus posited, in increasing degrees of complexity.

4.1 Labelled Graph Variants

These problem variants concern themselves with labelled graphs, that is relationships between named people.

Problem 1 *What is the number of ways a group of n people may be pairwise related or unrelated such that there are zero or more relationships within the group?*

This problem is dealt with in section 10.1.

Problem 2 *What is the number of ways a group of n people may be pairwise related or unrelated such that there is at least one relationship within the group?*

This problem is dealt with in section 10.2.

Problem 3 *What is the number of ways a group of n people may be pairwise related or unrelated such that there is at least one relationship within the group, and no single people?*

This problem is dealt with in section 10.3.

Problem 4 *What is the number of ways a group of n people may be pairwise related such every person is related, directly or indirectly, to every other person? This problem clearly subsumes the previous one.*

Table 1: The seven problems of section 4 and how they address the four questions of section 3.

Problem	Q1	Q2	Q3	Q4	Discussion
Tacit	Not empty	Singles	Not accounted for	People	In section 9
Problem 1	May be Empty	Singles	Connected or disjoint	People	In section 10.1
Problem 2	Not Empty	Singles	Connected or disjoint	People	In section 10.2
Problem 3	Not Empty	No singles	Connected or disjoint	People	In section 10.3
Problem 4	Not Empty	No singles	Connected only	People	In section 10.4
Problem 5	May be Empty	Singles	Connected or disjoint	Topologies	In section 10.5
Problem 6	Not Empty	No singles	Connected or disjoint	Topologies	In section 10.6
Problem 7	Not Empty	No singles	Connected only	Topologies	In section 10.7

4.2 Unlabelled Graph Variants

These problem variants concern themselves with unlabelled graphs, that is the *topology* of relationships. There are, naturally, fewer topologies than possible applications of those topologies to actual people. There is only one vee topology between three people, but given three *specific* people, there are three different vee configurations (A—B—C, B—A—C, A—C—B). These three labelled graphs are said to be *isomorphic*. Once the labels are removed, all three are topologically identical, leaving the ideal of a vee.

Problem 5 *How many different non-isomorphic ways are there for a group of n people to be pairwise related or unrelated such that there are zero or more relationships within the group?*

This problem is dealt with in section 10.5.

Problem 6 *How many different non-isomorphic ways are there for a group of n people to be pairwise related or unrelated such that there is at least one relationship within the group?*

This problem is dealt with in section 10.6.

Problem 7 *How many different non-isomorphic ways are there for a group of n people may be pairwise related such every person is related, directly or indirectly, to every other person?*

This problem is dealt with in section 10.7. The way each problem addresses each of the four questions (raised in section 3) is illustrated in table 1.

5 Injecting More Elegance into the Original Formula

Tacit et al.'s formula defines the number of possible 'relationship configurations' as follows:

$n=0$:						1					
$n=1$:					1		1				
$n=2$:				1		2		1			
$n=3$:			1		3		3		1		
$n=4$:		1		4		6		4		1	
$n=5$:	1		5		10		10		5		1

Figure 2: Pascal's triangle. The k-th term of the n-th row is equal to the binomial coefficient $\binom{n}{k-1}$.

$$t(n) = \left(\sum_{k=2}^{n-1} \frac{n!}{k!\,(n-k)!}\right) + 1, \quad \text{where } n, k \geq 2. \tag{1}$$

In Combinatorics, the binomial coefficient $\binom{n}{k}$, sometimes referred to as 'n choose k', enumerates the number of k-groupings that can be formed out of n objects ('combinations'). It is one of the many functions and sequences contained in Pascal's triangle, as illustrated in figure 2. It is defined as:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Noting that the summed term $n!/[k!\,(n-k)!]$ in equation 1 is the definition of the binomial coefficient $\binom{n}{k}$, the equation is simplified as:

$$t(n) = \sum_{k=2}^{n-1} \binom{n}{k} + 1, \quad \text{where } n, k \geq 2. \tag{2}$$

In this problem, the equation enumerates how many k-sized subgroups of people can be selected from a larger group of n people. The addition of the unit is explained as accounting for the case where all n people are related. Note, however, that this last so-called special case is equivalent to $\binom{n}{n}$ ('choose n people from n people'), and that $\binom{n}{n} = 1$. Thus, the same effect could be achieved with significantly more elegance by summing the binomial coefficient up to n, which obtains a simpler, more elegant form of this equation:

$$\begin{aligned} t(n) &= \sum_{k=2}^{n-1} \binom{n}{k} + \binom{n}{n} = \\ t(n) &= \sum_{k=2}^{n} \binom{n}{k},\ n \geq 2. \end{aligned} \tag{3}$$

This brings the equation to its most interesting form for the purpose of this paper. However, the equation can be simplified further. Note that $\binom{n}{k}$ is the $k+1$-th term of the n-th row of Pascal's triangle (figure 2). Also, recall or observe that the n-th row of Pascal's triangle sums up to 2^n. From this, we express $t(n)$ in terms of the sum from $k = 0$ to n, subtracting the excluded cases, $k = 0$ and $k = 1$, and eventually rewriting the sole remaining summation:

$$\begin{aligned} t(n) &= \sum_{k=2}^{n} \binom{n}{k} \\ &= \sum_{k=0}^{n} \binom{n}{k} - \sum_{k=0}^{1} \binom{n}{k} \\ &= \sum_{k=0}^{n} \binom{n}{k} - \binom{n}{1} - \binom{n}{0} \\ &= \sum_{k=0}^{n} \binom{n}{k} - \frac{n!}{1!\,(n-1)!} - \frac{n!}{0!\,(n-0)!} \\ &= \sum_{k=0}^{n} \binom{n}{k} - n - 1 \\ &= 2^n - n - 1,\ n \in \mathbb{N}, n \geq 2. \end{aligned} \tag{4}$$

This is trivial to verify against table 3, if not quite complex enough to put on popular merchandise.

The series generated by this function is $1, 4, 11, 26, 57, 120, 247, 502, 1013, \ldots$. It is catalogued in the Online Encyclopedia of Integer Sequences [Sloane et al, 2006] as part of sequence A000295 [Sloane et al, 2006a], with A000295 having zeroes for $n = 0$ and $n = 1$.

6 Signs of Incorrectness

Observing the penultimate form of the original equation, equation 3, four main issues become apparent.

1. The counting sometimes includes single people, for example for $n = 5$, when $k = 2$, the summed term yields the ways in which *one* couple can exist among five people, where three people are single.
2. The enumeration is unable to account for some relationships in groups greater than four people. A five-person group can contain a triangle and a couple that are mutually disjoint: by definition, no member of the triangle is connected to any member of the couple and, obviously, vice versa. This is overlooked.
3. These combinations lack topological nuances invaluable in differentiating between possible polyamorous relationship graphs. For example, for $n = 5$, when $k = 4$, there are $\binom{5}{4} = 5$ possible four-people groupings. The method fails to account for the *different* ways in which each of these four-person groups can be related *within the group*. Is everyone related to everyone else? Does one person have three relationships with the other three? Is it an expanded triangle where one person maintains an additional relationship? It will be shown that, for this example, there are 41 ways a group of 4 people can be pairwise related. This is illustrated in figure 4 (unshaded graphs). Only one of these is accounted for by this formula.
4. The rate of growth of equation 3 is $O(2^n)$, which is more akin to that of combinatorics problems. Intuition would expect answers to this problem to explode factorially.

7 Inconsistencies in Problem Description

It appears different encounters of this 'official mathematical equation' [Poly Tees, 2008] provide mutually conflicting problem descriptions — this naturally makes it difficult to gauge the intent of the question when only the answer is consistent. Here is a sample of such descriptions:

- The equation to count 'for any size group of people n how many possible relationship configurations (couples, triads, and so on) are possible within that group' (sic, Tacit et al [2006]). The original description, and the one on which this paper is based.
- From the Poly Tee website [Poly Tees, 2008]: 'the official mathmatical equation for finding the total number of possible relationships within a given poly relationship, assuming all people could be in a relationship with all others in the group.'

The original description seems to be nearer the truth, in that it *does* enumerate the total number of *single*[1],

[1] Single, since it fails to enumerate cases where mutually isolated subgraphs may be present, such as two unrelated triads in a group of six people.

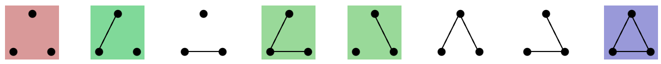


Figure 3: All possible three-vertex graphs. Red and green graphs have isolated vertices; green and blue graphs are enumerated by Tacit's formula.

fully-connected k-subgraphs of a labelled, undirected n-graph.

The latter description seems to indicate the original *intent* of the question, which seems to be corroborated by Tacit's write-up on the motivations of the derivation.

As it turns out, the formula is incorrect for both cases, although arriving at correct formulae for both cases is a daunting task.

8 Proof of Incorrectness

The incorrectness of $t(n)$ will be demonstrated below using various means.

It is also possible to visualise the incorrectness using figure 4. The figure displays all 64 possible labelled, undirected four-vertex graphs. Graphs shaded in red or green contain isolated vertices. The eleven graphs shaded green or blue are the ones enumerated by Tacit's equation.

As an additional demonstration, all possible 1,024 five-vertex graphs are provided in section 11. The disparity between the combinatorially-exploding wealth of possible graphs and those enumerated by $t(n)$ is quite obvious.

Proof by counter-example. For $n = 4$, the original formula yields:

$$\text{(from equation 1)} \quad t(4) = 2^4 - 4 - 1 = 11$$

All possible ways in which four people may be pairwise related or unrelated, id est all possible four-vertex labelled, undirected graphs, are illustrated in figure 4. Let us compare the seven previously described problems against the results yielded by equation 1. Counter-examples will thus be demonstrated.

- Problem 1: by inspection, there are 8×8 possible 4-vertex graphs (this is corroborated in section 10.1). This does not match the result of 11 above.
- Problem 2: all 64 graphs but one have at least one edge (relationship) in them, thus the answer to this is 63. Again, this does not match the original formula result.
- Problem 3: graphs in figure 4 shaded red or green are those with isolated vertices, id est those that involve single people. There are 23 such graphs, hence there are $64 - 23 = 41$ graphs that do *not* involve isolated vertices. Yet again, this does not equal 11.
- Problem 4: by examining the unshaded graphs in figure 4, it is obvious that there are exactly three unconnected graphs, id est graphs which contain multiple, mutually disjoint subgraphs, or involve multiple, mutually unrelated people. For clarity, those are repeated in figure 5. Thus, of the 41 graphs that do not involve isolated vertices, $41-3 = 38$ are connected graphs, satisfying the description

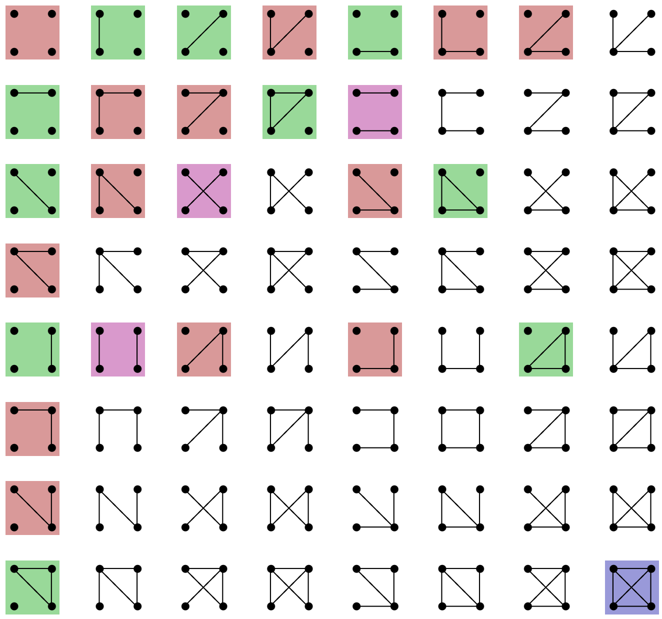


Figure 4: All possible labelled four-vertex graphs (in the interest of simplicity, the labelling itself is not shown). Red and green graphs have isolated vertices; green and blue graphs are enumerated by Tacit's formula; magenta graphs contain multiple mutually unrelated polyamorous groups. Note that many graphs representing what most conventions consider polyamorous relationships (for example, the top right graph, one person having three partners), are not considered as possible polyamorous 'relationship configurations' by Tacit's equation.

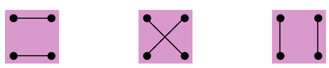


Figure 5: The three cases of two mutually disjoint relationships in 4-people groups, taken from figure 4, where they are shaded in magenta. Mutually disjoint relationships become increasingly more common as groups of five or more people are examined.

of this problem. Every vertex is directly or indirectly connected to every other vertex.

- Problem 5: by examining all graphs in figure 4, it is possible to locate eleven distinct non-isomorphic shapes (see table 2). This happens to match the original formula. By evaluating the formulae for both problems or examining the sequences themselves, however, it can be seen that this is a coincidence. One of the eleven isomorphic equivalence classes is the empty graph, which is expressly excluded from Tacit's formula, problem description, and common sense[2]. Also, their results do *not* match for $n > 4$.
- Problem 6: by examining table 2, it is trivial to count ten distinct shapes with one or more edges. Once more, this does not match what the original equation yields.
- Problem 7: finally, there are six distinct non-isomorphic connected topologies. On table 2, they are the unshaded ones, and the single, fully-connected topology shaded in blue.

By inspection, it is clear that none of the enumerations provided by the seven problem variants match that yielded by equation 1. However, it seems that Problem 5 most closely matches the intent of the original author, as values for $2 \leq n \leq 4$ coincide. It will, however, be demonstrated that this does not hold in the general case for $n > 4$.

Interestingly, subtracting 1 from the terms of the equation yields the sequence $0, 0, 3, 10, 25, 56, \ldots$, which *does* exist in Sloane et al [2006]. It is designated A097763 [Sloane et al, 2006g] and described as the 'number of different partitions of the set $\{1, 2, \ldots, n\}$ into an odd number of blocks such that each block contains at least 2 elements.'

This is obvious by inspecting the formula and its use of the binomial coefficient, which counts groupings but overlooks the relationships between members of the group.

By identifying this as a set-theory sequence, the earlier assumption that the formula fails to account for topological differences between sub-groups is partially corroborated.

Proof by reduction to a known problem.

Problem 8 *Assume a special interest group* $A = \{x1, x2, \ldots, x_n\}$ *of* $n \geq 2$ *people in need of forming a committee of some of its members. How many different committees of two or more people could be formed?*

[2]It is difficult to claim four *unrelated* people are in a polyamorous group.

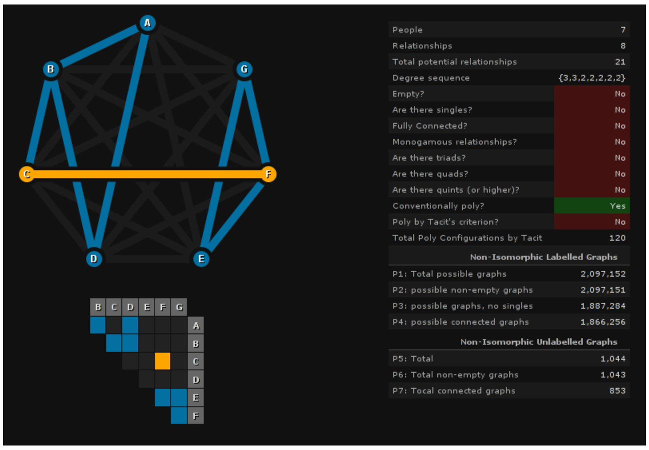


Figure 6: The polyamorous graph explorer, a visualisation and exploration tool for polyamorous relationships.

This is a text-book problem in combinatorics. The total number of two-person committees is $\binom{n}{2}$. The total number of three-person committees is $\binom{n}{3}$. There is only one n-person committee, as $\binom{n}{n} = 1$. Summing these up,

$$\sum_{k=2}^{n} \binom{n}{k} = 2^n - n - 1, \; n \geq 2, \tag{5}$$

and this is identical to equation 3. Now, note that a committee B is a subset $B \subseteq A$ of the original group of people, and that a set, by definition, contains no information about potential relationships between its members. It is thus clear that this formula does not answer the original problem.

Proof by Visualisation. It is trivial to prove the incorrectness of $t(n)$ by examining the visualised 3-, 4-, and 5-graphs in figure 3, figure 4 and section 11 respectively.

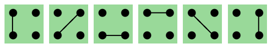
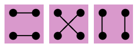
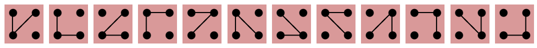
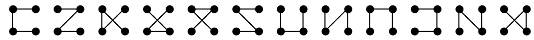
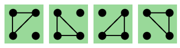
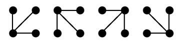
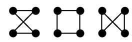
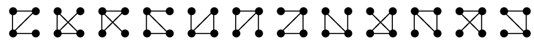
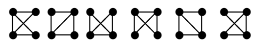
The green and blue-shaded graphs represent those enumerated by $t(n)$. Red-shaded graphs contain isolated vertices. Red graphs are not enumerated by $t(n)$. Unshaded graphs represent connected graphs not enumerated by $t(n)$.

Inspection of the graph visualisation demonstrates that most graphs identifying polyamorous relationships are not enumerated by $t(n)$ — it only enumerates particular types of polyamorous relationships.

The author has prepared an interactive web-based tool intended to explore such graphs. The tool allows the user to construct arbitrary relationship graphs and to see if Tacit's equation enumerates them, as well as to glean various information about the graphs' topology and combinatorics.

The tool may be found at http://www.bedroomlan.org/tools/polyamory-graph-explorer. It requires a recent web browser capable of displaying HTML Canvas elements

Table 2: Isomorphism in 4-vertex graphs. The 64 labelled graphs are all topologically isomorphic in eleven groups. If the labels were removed, the graphs on each row would all be identical.

Degree seq	Edges	Isomorphic graphs	
$\{0, 0, 0, 0\}$	0	1	
$\{1, 1, 0, 0\}$	1	6	
$\{1, 1, 1, 1\}$	2	3	
$\{2, 1, 1, 0\}$	2	12	
$\{2, 2, 1, 1\}$	3	12	
$\{2, 2, 2, 0\}$	3	4	
$\{3, 1, 1, 1\}$	3	4	
$\{2, 2, 2, 2\}$	4	3	
$\{3, 2, 2, 1\}$	4	12	
$\{3, 3, 2, 2\}$	5	6	
$\{3, 3, 3, 3\}$	6	1	

and Javascript. Please refer to figure 6 for an example image of the tool at work.

9 What *Does* Tacit's Formula Enumerate?

If Tacit's Formula fails to enumerate correctly any of the seven problems described above, what exactly *does* it enumerate? The question is a valid one.

The result of $t(n)$ appears to be the total number of sub-groups between 2 and n people that may be selected from an n-person group.

In a derivation of an answer to any of the intended original problems, this formula would be the first step. First, count the number of different ways we can select people for relationships. Then, count how many different ways each of these groups of people may be related in, and multiply the two. This is a common technique in simple combinatorics problems, but fails in a number of ways here.

Of all the possible different ways k people in an n people group may be connected, $t(n)$ only accounts for one. By convention, in this paper, this way is the fully-connected subgraph. The author has encountered an explanation of the formula[3] which corroborates this: if persons A and B are in a relationship, and persons B and C are also in a relationships, then, since this is a polyamorous group, people A and C will know of each other, and thus also have a relationship (even simply a friendly one). This sheds some light on the line of thinking that may have led to the equation's strange result. Given Tacit's active involvement in polyamory, however, it is difficult to accept this as an explanation. Polyamory deals with romantic relationships, after all — not friendly ones. In this context, persons B and C are only in a relationship if they are in a full-fledged romantic relationship. Knowing about each other's existence doesn't mandate that they have ever met, or know each other personally.]

For lack of more detailed, consistent discussion, however, $t(n)$ may be defined as: the total number of fully-connected k-subgraphs $(2 \leq k \leq n)$ in a graph of n nodes (people), provided the graph includes no mutually disjoint subgraphs and zero or more isolated vertices.

9.1 Similar Integer Sequences in the Literature

For $n \geq 2$, the sequence generated by $t(n)$ is $1, 4, 11, 26, 57, 120, 247, 502, 1013, \ldots$. This is found within Sloane's Integer Sequence A000295 [Sloane et al, 2006a] (Column 2 of Euler's Triangle), starting with the third element of that sequence (A000295 yields 0 for $n = 0$ and

[3] A reference to which is not currently available.

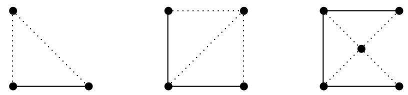


Figure 7: New people joining existing groups. The n-th person to join a group may attach in any combination of 1 to $n-1$ ways, shown above as dotted edges.

$n = 1$, rendering $0, 0, 1, 4, 11, \ldots$). The entry for this sequence lists $t(n)$ explicitly as the generating formula for this sequence, and provides a number of possible explanations of what the sequence enumerates, including the chromatic invariant of Prism Graph P_n.

This sequence matches exactly the intent of $t(n)$: without any people ($n = 0$), there are exactly zero polyamorous relationships possible; with one person ($n = 1$), there are likewise no relationships possible, so $t(n)$ may be generalised to a[4] form of:

$$t(n) = \begin{cases} 0 \le n < 2 : 0 \\ 2 \le n : n^2 - n - 1 \end{cases} \tag{6}$$

The exact sequence $1, 4, 11, 26, 57, 120, 247, 502, 1013, \ldots$ is found as Sloane's A125128 [Sloane et al, 2006h].

Interestingly, the final Sloane integer sequence that subsumes that of $t(n)$ is A130103 [Sloane et al, 2006i]. It runs $0, 1, 1, 4, 11, 26, 57, 120, 247, 502, 1013, \ldots$ and is defined as the expansion of $e^{2x} - (1+x)e^x + 1$.

10 Correct Formulae

The discussion below provides correct formulae for four of the seven previously described variants of the problem at hand. Calculations for the other three exist in the literature, but are not detailed in this paper for reasons of brevity.

10.1 Problem 1

The problem asks 'what is the number of ways a group of n people may be pairwise related or unrelated such that there are zero or more relationships within the group?'

To calculate this, the maximum number of edges in an n-vertex graph is needed. From graph theory, this is $\frac{1}{2}n(n-1)$, or $\binom{n}{2}$, id est the triangular number that is the sum of the n-term arithmetic sequence $1+2+3+\cdots+n$. This may be visualised by realising that, in an n-person polyamorous group, a new, $n+1$-th person could form any combination of one to n relationships. This is illustrated in figure 7.

An edge can either exist or not, which gives it two possibilities. For $n = 2$, there is one edge, hence two possible graphs. For $n = 3$ there are $1 + 2 = 3$ edges, hence $2 \times 2 \times 2 = 2^3 = 8$ possible graphs. For $n = 4$, there are $1 + 2 + 3 = 6$ possible edges, hence $2^6 = 64$ possible graphs, and so on. This makes the formula for the answer to this problem:

$$a(n) = 2^{\binom{n}{2}}, \text{ or} \tag{7}$$

$$a(n) = 2^{\frac{1}{2}n(n-1)}. \tag{8}$$

The integer sequence $1, 2, 8, 64, 1024, \ldots$ is Sloane's A006125, and is a well-known sequence in graph theory.

10.2 Problem 2

This problem is essentially identical to Problem 1, with one exception: it postulates at least one relationship must exist in the group.

The answer is trivial, as there is exactly one n-vertex graph with no edges, the empty graph. As such, the answer $b(n)$ to this problem is:

$$\text{(from equation 7)} \quad b(n) = 2^{\binom{n}{2}} - 1. \tag{9}$$

This is trivial to visualise using figure 3 and figure 4.

10.3 Problem 3

This problem is much less trivial. It involves the number of n-vertex graphs that have no isolated vertices (or no single people).

The answer is difficult to arrive at, but exists in the literature. A computer programme may be constructed to enumerate graphs that conform to this rule. The sequence arrived at, the first three terms of which may be gleaned by examining figures figure 1, figure 3 and figure 4, is $0, 1, 4, 41, 768, 27449, \ldots$. This may be cross-referenced as Sloane's A006129 [Sloane et al, 2006e], and is defined as a somewhat daunting $c(n)$ where:

$$c(n) = \sum_{k=0}^{n} -1^{n-k} \binom{n}{k} 2^{\binom{k}{2}}. \tag{10}$$

Obtaining this formula is a non-trivial task. The exact details are outside the scope of this document. The enthusiastic student of discrete mathematics may want to peruse Sloane et al [2006e], Harary [1994].

The author expects this formula will be of the most use to people reading the original article, as it most closely matches the consensus arrived about the characteristics of the original problem.

[4] Much more merchandise-friendly — if still incorrect.

Table 3: The sequences generated by the original formula, equation 1, and those for the non-trivial problem variants. The trivial sequences for problems 2 and 6 are omitted, as they are equal to the sequences for problem 1 minus 1 (see equation 9), and problem 5 minus 1 (equation 17) respectively.

n	$t(n)$	Prob. 1	Prob. 3	Prob. 4	Prob. 5	Prob. 7
2	1	2	1	1	2	1
3	4	8	4	4	4	2
4	11	64	41	38	11	6
5	26	1,024	768	728	34	21
6	57	32,768	27,449	26,704	156	112
7	120	2,097,152	1,887,284	1,866,256	1,044	853
8	247	268,435,456	252,522,481	251,548,592	12,346	11,117
9	502	68,719,476,736	66,376,424,160	66,296,291,072	274,668	261,080
10	1,013	35,184,372,088,832	34,509,011,894,545	34,496,488,594,816	12,005,168	11,716,571
Sequence:		Sloane A006125	A006129	A001187	A000088	A001349

10.4 Problem 4

Problem 4 is deceptive, in that it looks quite similar to Problem 3, but is in fact quite drastically different. It pertains to the enumeration of all connected graphs of n vertices. The resultant sequence is similar in order of magnitude to that of Problem 3, but naturally different. Deriving the formula is highly non-trivial, but a derivation already exists in Wilf [1990] by way of exponential generating functions and logarithmic transforms.

The number of connected, labelled, undirected graphs of n vertices is connected to the power series $f(d_n;\, x)$ with exponential generating function

$$\text{egf} \quad = \quad \sum_{k=0}^{n} \frac{2^{\binom{k}{2}}}{k!}\, x^k. \qquad (11)$$

Transforming this logarithmically provides:

$$f(d_n;\, x) \quad = \quad 1 + \log\left[\sum_{k=0}^{n} \frac{2^{\binom{k}{2}}}{k!}\, x^k\right]. \qquad (12)$$

This may be used to calculate the number of connected, labelled graphs by obtaining the polynomial $f(d_n;\, x)$ and applying the its n-th order coefficient $f(d_n;\, x)_n$ to the following equation:

$$d(n) \quad = \quad n!\, f(d_n;\, x)_n. \qquad (13)$$

For $n = 1, 2, \ldots$, this equation yields the integer sequence $1, 1, 4, 38, 728, 26704, \ldots$, designated Sloane's A001187 [Sloane et al, 2006f].

10.5 Problem 5

The formula that enumerates all possible non-isomorphic, unlabelled, undirected graphs with a given number of vertices exists in the literature, but is not particularly trivial. Its complexity vindicates the difficulty seen by Tacit et al [2006] in arriving at the original result, no matter its correctness.

It is recommended that the interested reader study Weisstein [2006] to obtain, by an application of the Pólya enumeration theorem [Pólya, 1937], a rather expansive counting polynomial $g_p(x)$,

$$g_p(x) = \sum_k g_{pq} x^k, \qquad (14)$$

where $g_p q$ is the number of distinct graphs with p vertices and q edges. The actual form of the polynomial is found to be

$$g_p(x) = p!\, Z\left(S_p^{(2)},\, 1 + x\right), \qquad (15)$$

where $S_p^{(2)}$ is the pair group that acts on the 2-subsets of $\{1, 2, \ldots, p\}$, and is given by

$$Z(S_p^{(2)}) = \frac{1}{p!} \sum_{(j)} h_{\mathbf{j}} \prod_{n=0}^{\lfloor \frac{p-1}{2} \rfloor} a_{2n+1}^{n j_{2n+1} + (2n+1)\binom{j_{2n+1}}{2}}$$
$$\prod_{n=1}^{\lfloor \frac{p}{2} \rfloor} \left[(a_n a_{2n})^{n-1}\right]^{j_{2n}} a_{2n}^{2n\binom{j_{2n}}{2}} \prod_{q=1}^{p} \prod_{r=q_1}^{p} a_{\mathrm{lcm}(q,r)}^{j_q j_r \gcd(q,r)}, \qquad (16)$$

where $\lfloor \cdot \rfloor$ is the *floor* function, $\mathrm{lcm}(x, y)$ is the lowest common multiple of x and y, $\gcd(x, y)$ is the greatest common divisor of x and y, the sum (j) is over all exponent vectors of the cycle index $Z(S_p)$ of the symmetric group S_p, and $h_{\mathbf{j}}$ is the coefficient of the term with exponent vector $\mathbf{j}_p$ in $Z(S_p)$ [Weisstein, 2006, Harary, 1994].

Example expansions of $Z(S_p^{(2)})$ are:

$$
\begin{aligned}
Z\left(S_1^{(2)}\right) &= 1 \\
Z\left(S_2^{(2)}\right) &= a_1 \\
Z\left(S_3^{(2)}\right) &= \frac{a_1^3}{6} + \frac{a_1 a_2}{2} + \frac{a_3}{3} \\
Z\left(S_4^{(2)}\right) &= \frac{a_1^6}{24} + \frac{3a_1^2 a_2^2}{8} + \frac{a_3^2}{3} + \frac{a_2 a_4}{4} \\
Z\left(S_5^{(2)}\right) &= \frac{a_1^{10}}{120} + \frac{a_1^4 a_2^3}{12} + \frac{a_1^2 a_2^4}{8} + \frac{a_1 a_3^3}{6} + \frac{a_2 a_4^2}{4} + \frac{a_5^5}{5}
\end{aligned}
$$

Normalising by $p!$ and letting $a_i = 1 + x^i$ gives $g_p(x)$, such as:

$$
\begin{aligned}
g_2(x) &= 1 + x \\
g_3(x) &= 1 + x + x^2 + x^3 \\
g_4(x) &= 1 + x + 2x^2 + 3x^3 + 2x^4 + x^5 + x^6 \\
g_5(x) &= 1 + x + 2x^2 + 4x^3 + 6x^4 + 6x^5 + 6x^6 + 4x^7 + \\
&\quad +2x^8 + x^9 + x^{10}
\end{aligned}
$$

The sum of coefficients of g_n above gives the number of non-isomorphic unlabelled graphs of n vertices. Thus, given $x = 1$ and evaluating the polynomials, the sequence $g_n(1)$ for $2 \leq n \leq 5$ is $2, 4, 11, 34$. This sequence is provided in table 3 for comparative purposes. It can be found in Sloane et al [2006] as A000088 [Sloane et al, 2006b].

10.6 Problem 6

The number of non-isomorphic n-vertex graphs with at least one edge is equal to the number of all non-isomorphic n-vertex graphs minus the number of n-vertex graphs with zero edges. That number is always one, thus:

$$f(n) = g_n - 1 \tag{17}$$

The full derivation of $f(n)$ is purposefully excluded from this document.

10.7 Problem 7

The number of connected, unlabelled, undirected graphs of n-vertices is again given by Sloane et al [2006] as the inverse Euler transform of the sequence for Problem 5. The sequence itself is provided in table 3 for comparative purposes, and can also be found in Sloane et al [2006] as A001349 [Sloane et al, 2006c].

11 Conclusion

In a LiveJournal post, Tacit et al [2006] proposed an unfortunately erroneous, and somewhat under-specified equation to enumerate the 'total number of possible relationship configurations' for any group of polyamorous people.

This document explained the potential ambiguities of this approach, and formulated seven problem variants approaching the spirit of the original problem. The underlying domain was shown to be reducible to graph theory. Equations from graph theory were obtained to enumerate the different labelled, undirected graphs described by each of the problem variants, and values for these equations were calculated by computer programme and tabulated for the convenience of the reader.

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A Five-Vertex Graphs

The following three pages display all 1,024 possible five vertex, labelled, undirected graphs (ten possible edges, 2^{10} possible graphs). As with figure 4, graphs shaded red or green have isolated vertices. Green and blue graphs are graphs enumerated by Tacit's equation.

Magenta-shaded graphs are connected graphs with isolated subgraphs (multiple, mutually unrelated polyamorous groups represented in the same group of people).

The remaining unshaded graphs are connected without isolated subgraphs, yet not enumerted by Tacit's equation.

The sparsity of green and blue-shaded graphs demonstrates the inability of $t(n)$ to account for all possible ways in which even as few as five people may be pairwise related.

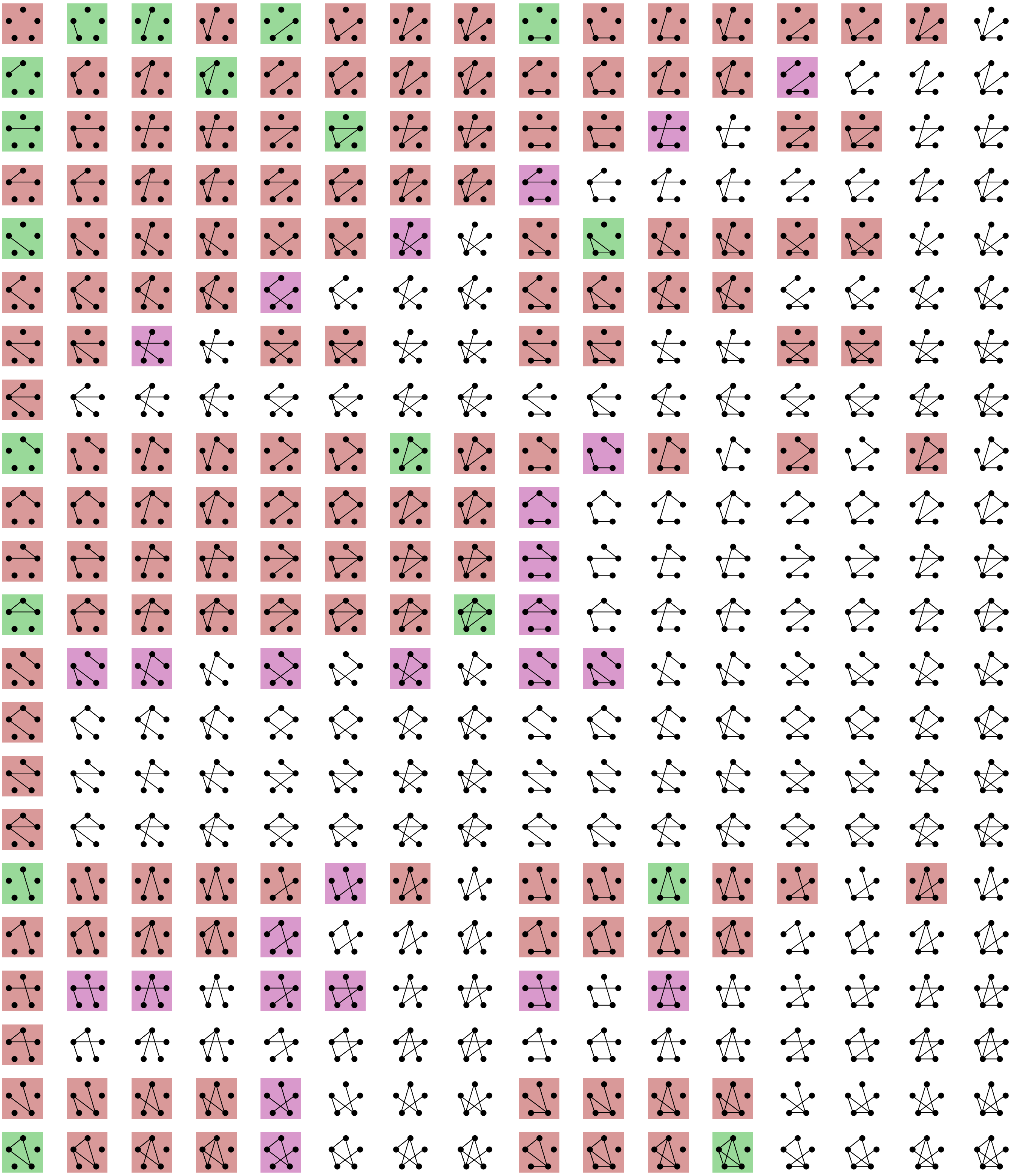

Figure 8: All possible labelled five-vertex graphs (part one of four).

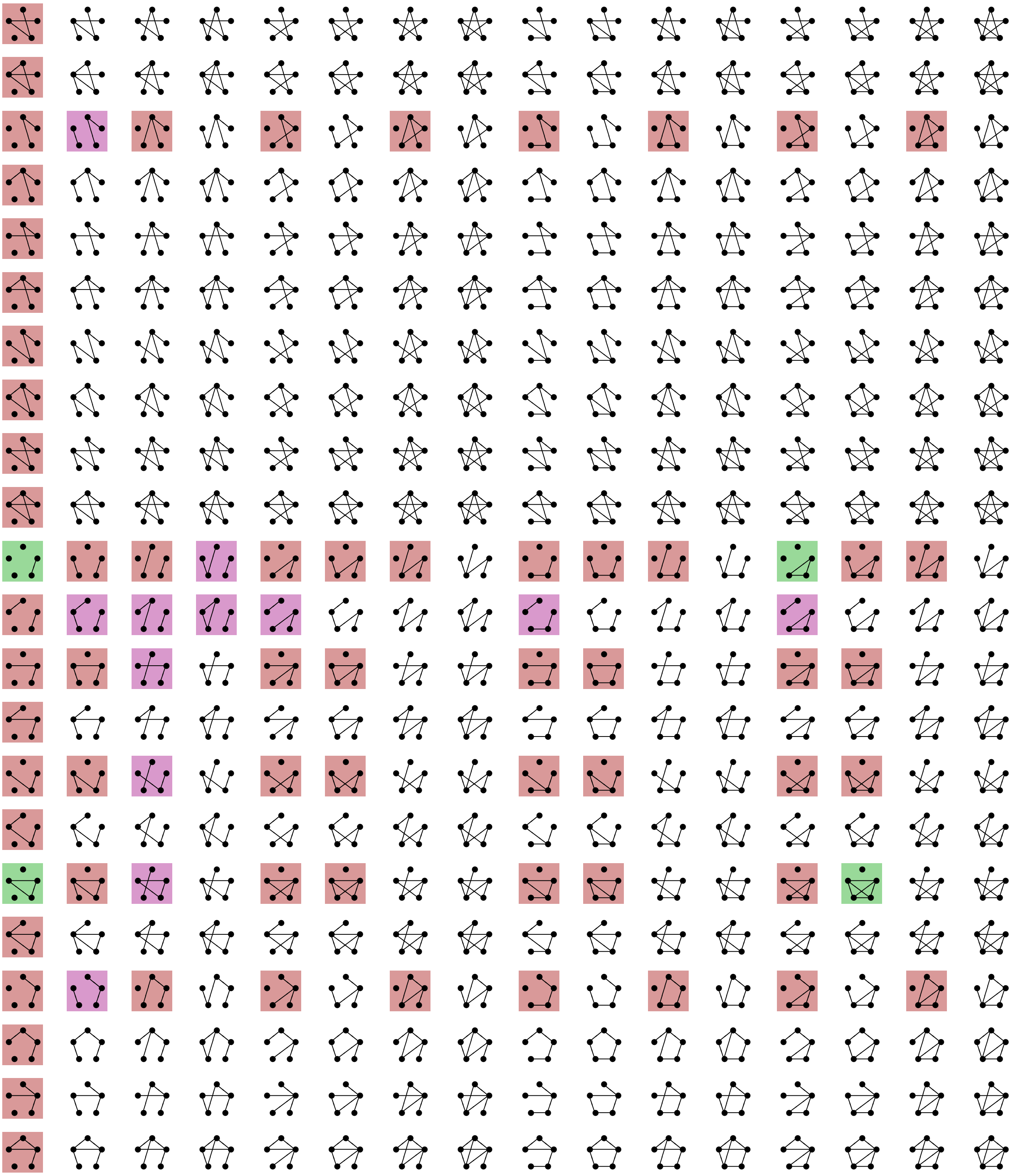

Figure 9: All possible labelled five-vertex graphs (part two of four).

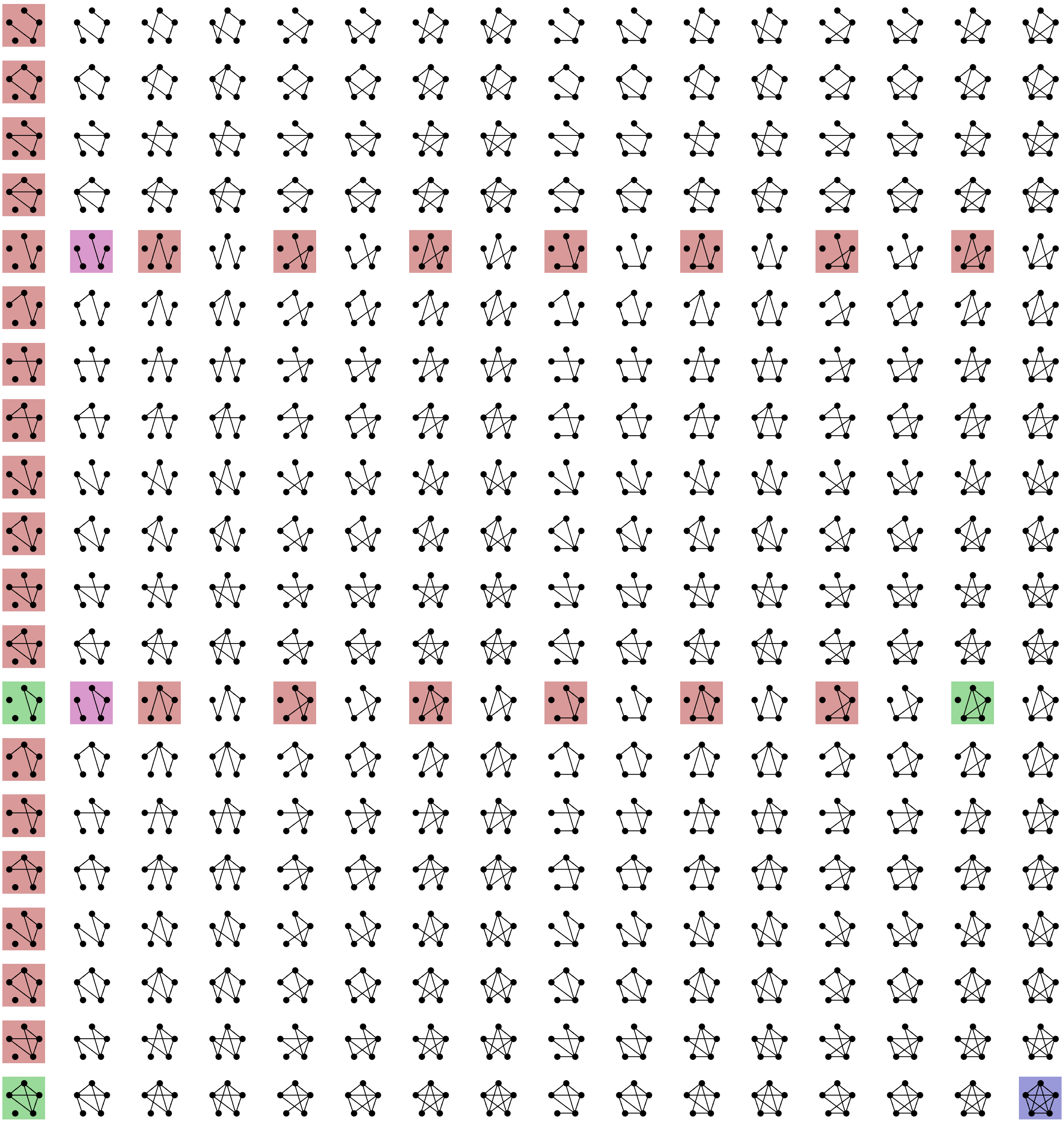

Figure 10: All possible labelled five-vertex graphs (part three of four).